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Determinants of oil futures prices and convenience yields

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Commodity futures prices are usually modelled using affine term structure spot price models with latent factors extracted from the data. However, very little research to date has considered the question – What are the economic drivers behind the calibrated latent factors? This paper addresses this question in the context of a three-factor – short-, medium- and long-term – model for crude oil spot prices by studying the relations between these factors and appropriate economic variables. An affine combination of the short- and medium-term factors is identified as the (instantaneous) convenience yield. Estimating a structural vector auto-regression model we find that the short-term factor mainly relates to demand variables in the physical markets and to trading variables in the futures markets (such as the net short position of commercial hedgers), the medium-term factor relates to business cycles, demand and trading variables, and the long-term factor relates mainly to financial factors.

Keywords: Oil futures; Futures term structure; Theory of storage; Theory of normal backwardation; Kalman filter; SVAR

1. Introduction

Commodity futures exchanges arose in the late 19th and early 20th centuries to allow the forward contract mitigation of cyclical supply/demand imbalances between agricultural producers and consumers while limiting speculation through requiring the posting of margin on futures positions. In the 21st century, expanded to a wide range of commodities and related services and increasingly electronic, they continue to serve their original purpose. But they are also vital to the forecasting of spot prices by large resource producers for management evaluation of project alternatives and investment opportunities over very long-term horizons. Global resource firms may or may not hedge their physical activities in the futures markets, but increasingly they have come to see that sophisticated price forecasting is a prerequisite to the use of real option techniques for forward planning and risk management of ongoing operations.

However, for use in exploration, acquisition evaluation, or project development and risk management, senior management cannot be content with reduced form 'black box' price forecasting methods devoid of an economic understanding of the commodity markets involved. Focussing on crude oil prices, this paper attempts to meet these stringent managerial criteria by specifying a three-factor spot price model for oil and studying its relationship to different economic variables, which includes financial variables (such as SP500 returns, US dollar returns, etc.), business cycle variables (such as the business cycle coincident index), demand variables (such as inventory and the heating-crude oil spread) and trading variables (such as futures open interest growth and hedging pressure). We hope to contribute to an understanding of the relationships between oil prices, physical inventory management, financial hedging and speculation. Although this paper treats the oil markets, the model treated here may be applied to a wider range of commodities, upon which we are currently engaged (Dempster and Tang 2011).

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Commodities are real assets and so their prices should be influenced by their supply and demand. The *theory of storage* (Kaldor 1939, Working 1949, Brennan 1958) sees optimal inventory management as the main determinant of commodity prices. But since commodities are traded through futures contracts, financial markets and trading behaviour will also influence the commodity prices and term structure, as noted by Keynes (1930) in his *theory of normal backwardation*. More recently, Bailey and Chan (1993) showed that financial factors such as the spread between BAA and AAA bonds can influence the convenience yield of many commodities. In this paper, we shall examine the impact both of supply, demand and business variables and of financial and trading factors on the movement and shape of the oil futures term structure.

Both producers and consumers wish to make forecasts for long-term planning and investment decisions since commodity prices represent their output revenues and input costs, respectively. Various authors have expressed differing opinions on the long-term evolution of commodity prices. Most (see, e.g., Cuddington and Urzua (1987) and Gersovitz and Paxson (1990)) believe that commodity prices are non-stationary since it is hard statistically to reject the most parsimonious geometric random walk model using historical time series data. Cashin *et al.* (2000) have shown that shocks to commodity prices are typically long lasting, while Grilli and Yang (1988) found that real primary commodity prices have a trend of about 0.5% a year using a dataset from 1900 to 1986. Schwartz and Smith (2000) use *geometric Brownian motion* (GBM) to model such long-term behaviour because of the ability of GBM to capture trend and the persistency of shocks. However, Bessembinder *et al.* (1995) discovered strong mean reversion in commodity log spot prices, suggesting that a *geometric Ornstein Uhlenbeck* (GOU) process might be more appropriate. Although not pointed out explicitly, Casassus and Collin-Dufresne (2005) use a mean-reverting process to model log spot commodity prices. Geman and Nguyen (2005) use a mean-reverting log spot price with stochastic mean and stochastic volatility to model soybean futures prices.† Many economists believe that commodity prices in the medium term are closely related to the business cycle (e.g., Fama and French (1988)), which is usually considered to be a mean-reverting process. Short-term swings in commodity prices have substantial impacts for many speculators and short-term strategic investors and the short-term factors driving commodity prices are usually also considered to be mean-reverting (e.g., Schwartz and Smith (2000)).

In this paper we use a mean-reverting process to model the short-term factor in our three-factor (log) spot price model. However, it is not appropriate to model short-term influences using only a *single* mean-reverting factor, since one factor alone cannot adequately model the complicated short- to medium-term behaviour of commodity prices. This suggests that two mean-reverting factors—one short- and one medium-term—are needed to model price movements.‡ Our medium-term factor captures business cycles and long-term demand in the global economy, while a long-term GBM factor captures trend-related persistent shocks, such as technology growth, long-term supply through the discovery of new resources, etc. Intuitively, the time scale of the short-term factor is several months, that of the medium-term factor 1 to 2 years and that of the long-term factor decades or even longer. The three-factor model treated here nests several other models, including those of Gibson and Schwartz (1990) and Schwartz and Smith (2000), which are equivalent. Because of oil's importance to the global economy, and the ease of obtaining inventory data for it and relevant economic variables, we use crude oil futures to illustrate our model's development and to examine the various intuitions presented briefly above.

After developing the model§ in state space form, we use the Kalman filter to obtain the estimated historical paths of the three latent factors from observations of oil futures prices. We then perform a *structural vector auto-regression* (SVAR) analysis involving the three estimated factor paths and the historical paths of several economic variables, including financial, business-cycle, fundamental and trading variables. We find that financial variables mainly affect the long-term p factor and the business cycle variable influences mainly the medium-term y factor. The demand variables affect both x and y factors, and higher net demand results in both a higher short-term x factor (deeper short-end backwardation) and a higher y factor (deeper long-end backwardation). The trading variables also influence both x and y factors, and more intensive trading and stronger hedging pressure result in higher factor levels.

The paper is organized as follows. Section 2 examines several features of WTI crude oil prices and develops a detailed motivation for a three-factor spot price model. Section 3 presents such a model and explains its relationship to earlier models. Section 4 examines the relationship between the three estimated latent factors and several economic variables. Section 5 concludes.

2. Oil price features

In this section we characterize features of crude oil futures prices and their evolution.

2.1. Term structure of oil futures open interest

Weekly oil futures prices and open interest for WTI crude oil (CL) traded on the New York Mercantile Exchange (NYMEX) were obtained from 1986.06 to 2010.12 from

† Using soybean inventory data, Geman and Nguyen (2005) also show that soybean futures return volatility is negatively related to soybean inventory (or positively related to 'scarcity', the reciprocal of inventory), which is consistent with the theory of storage.
‡ The medium-term factor should obviously revert to its mean more slowly than the short-term factor.
§ It is an affine futures term structure model for the log spot price in the $A_0(3)$ form of Dai and Singleton (2000).

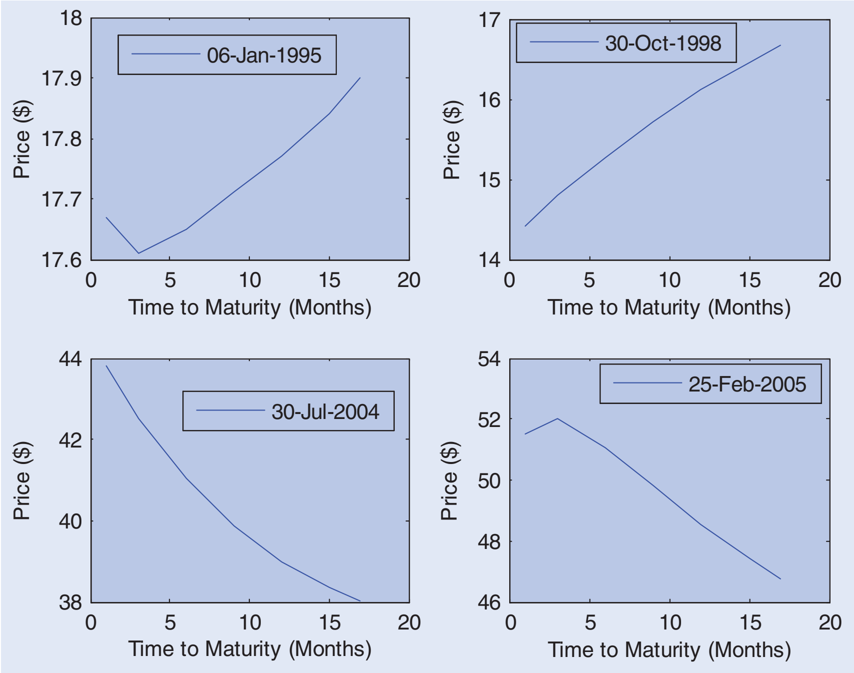

Figure 1. The oil futures term structure.

Pinnacle Data Corp. The times to maturity of these futures contracts range from several days to about 17 months† (the first to the seventeenth contract). Figure 1 shows futures term structure shapes commonly observed in the market with their observation dates. At the long end we can see both *contango* (top two diagrams) and *backwardation* (bottom two diagrams) term structures, while at the short end we see U and hump shapes. Thus the short end of the term structure appears to be more volatile and is not necessarily conformal with the long end.

Futures prices are discovered through trading, so to investigate their short-term behaviour in figure 2 we plot the futures *open interest,* i.e. the total number of futures contracts that have not expired, or been fulfilled by delivery, against their times to maturity. The figure demonstrates that the main oil trading activities are concentrated in the one month futures contract. Its open interest is much larger than the value obtained from an exponential function fit to open interest at other maturities.‡ This large open interest in the nearby contract indicates a uniquely high liquidity which can result in different behaviour of short-term oil futures prices. Furthermore, investors and hedgers all prefer to use short-term futures instead of long-term futures and thus contribute to this behaviour. For example, passive commodity index investors tend to invest in short-term rather than long-term futures (for details, refer to Tang and Xiong (2010)). As shown by Culp and Miller (1995), Mellon and Parsons (1995) and Brennan and Crew (1997), hedgers employ short-term futures to hedge longer-term obligations.

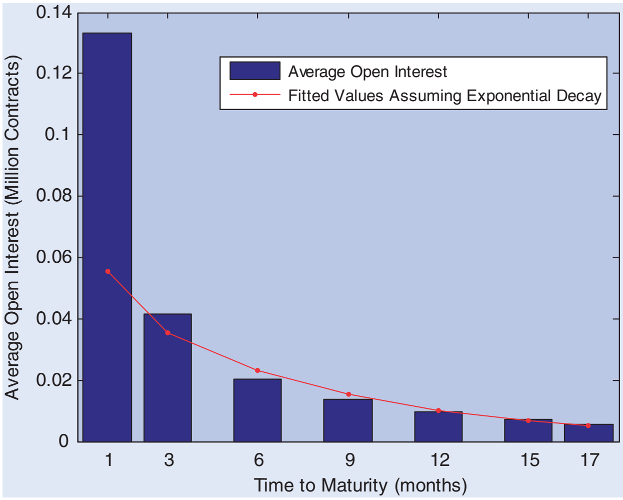

Figure 2. Average open interest vs. time to maturity.

2.2. *Single-factor convenience yield models*

In previous research, nearly all researchers have used a single factor to model *convenience yield,* which is the commodity equivalent of equity dividend yield representing the opportunity return to physical ownership of the

† The open interest for futures with time to maturity longer than 17 months is very small (see figure 4). Also, in the earlier part of our dataset, futures prices with maturity longer than this are not available.

‡ We fit the open interest frequency by fitting $P_\tau = a\exp(b\tau)$ to it, where P_τ is the (time) average open interest for futures contracts with time to maturity τ.

Table 1. Model errors for futures with different times to maturity.

Maturity (months)	Gibson–Schwartz two-factor model	Schwartz–Smith two-factor model	Schwartz three-factor model
1	***0.043***	***0.042***	***0.045***
5	0.006	0.006	0.007
9	0.003	0.003	0.003
13	0	0	0
17	0.004	0.004	0.004

commodity. Among such models, the Gibson–Schwartz (1990) two-factor model, the Schwartz–Smith (2000) two-factor model and the Schwartz three-factor model are commonly used.† We use the Schwartz–Smith model here to investigate whether or not one factor is enough to model convenience yields.

2.2.1. Short term pricing errors. Table 1 contains the log pricing errors‡ from Schwartz (1997, p. 939) and Schwartz–Smith (2000, p. 903), where the short-term (1 month) futures prices have a noticeably larger error than the others.§ This is consistent with the hypothesis that short-term futures price movements are often not conformal with longer-term futures price movements and thus a *separate* factor is needed to model short-term futures price movements.

2.2.2. Convenience yields inferred from the Schwartz–Smith (2000) model. In the Gibson–Schwartz model, log futures prices are expressed as an affine combination of two latent factors: the log spot price and the convenience yield. Thus given parameter estimates for the model the convenience yield and spot price can be backed out using any two futures prices. We re-estimate the Gibson–Schwartz model using our own dataset and in figure 3 plot the evolution of the model convenience yields inferred from 1 and 3 month futures and 15 and 17 month futures and their difference. It is clear that these two estimates of convenience yield are *strongly inconsistent*. The unconditional standard deviation of the convenience yield implied from 1 and 3 month futures is 26.3% per annum, while that from 15 and 17 months is 34.7% p.a. The unconditional standard deviation of the difference between these two convenience yields is surprisingly large, about 33.6% p.a.¶ The differences between these two convenience yield maturities should therefore not be overlooked, but should instead be modelled using a new short-term factor.

2.2.3. Principal component analysis on convenience yields. As we have seen, one factor is not enough to model the convenience yield; in this section we test how many factors are actually needed using a *principal component analysis* (PCA). Since convenience yield is not directly observable, we infer the implied convenience yield $\delta(t,T)$ from commodity futures prices and interest rates using

$$\begin{aligned}\delta(t,T) &= r_t - \frac{\ln(F(t,T)) - \ln S_t}{T-t}\\ &\approx r_t - \frac{\ln(F(t,T)) - \ln(F(t,T_0))}{T-T_0},\end{aligned} \tag{1}$$

where T_0 is the time to maturity of nearby futures contracts, T is the time to maturity of futures with a relatively longer horizon and r_t is the instantaneous rate corresponding to the three month LIBOR rate. We use nearby and 3, 6, 9, 12, 15, 17 month futures (i.e. $T=3$, 6, 9, 12, 15, 17) to calculate six time series of implied convenience yield over our data period and then perform a PCA on them. Table 2 shows the variance explained by each factor. Clearly, two factors can explain more than 98% of the overall variance of the convenience yield, so that a two-factor model is good enough to catch its behaviour.

In the sequel we use two factors to model convenience yield, one short term and one for a longer medium-term horizon. The short-term factor should correct both the large pricing error of short-term contracts in two-factor models and the mismatch arising with these models of implied convenience yields backed out from short- and longer-maturity futures. Adding a long-term factor, the resulting three-factor model can capture the different shapes of the futures term structure shown in figure 2.

3. Three-factor model statement

We begin by modelling the dynamics of the log spot oil price $\mathbf{G}$ in terms of convenience yield using two factors.

† The two factors in the Gibson–Schwartz (1990) model are (log) spot price and convenience yield; the Schwartz three-factor model is an extension of the Gibson–Schwartz (1990) model with an additional stochastic interest rate factor; the Schwartz–Smith (2000) model is equivalent to the Gibson–Schwartz model.

‡ The *log pricing error* is defined as the standard deviation of the difference of the market and the model log futures prices. The models are calibrated and the futures prices calculated by the methods of sections 3 and 4 of this paper (see the original references for details).

§ When we estimate the Schwartz–Smith (2000) model using our dataset we find a similar phenomenon.

¶ Note that the implied convenience yield during the financial crisis fluctuates widely.

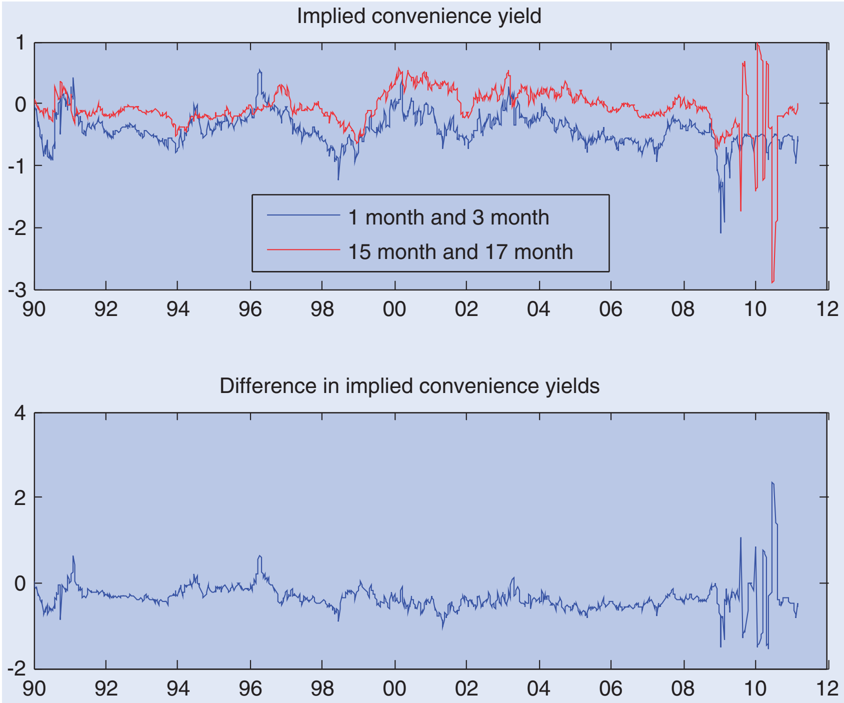

Figure 3. Implied convenience yields and their difference for the two-factor Gibson and Schwartz model.

Table 2. The principal components of the implied convenience yield.

Component	Variance explained (%)
First	92.05
Second	5.99
Third	1.33
Fourth	0.39
Fifth	0.22
Sixth	0.02

3.1. Dynamics of spot prices

In the *market* (physical) *measure* the system is given by†

$$d\mathbf{G}_t = \left(r^f + \lambda_G - \delta_t - \gamma_t - \frac{1}{2}\sigma_G^2\right)dt + \sigma_G d\mathbf{W}_\mathbf{G}, \tag{2}$$

$$d\boldsymbol{\delta}_t = k_\delta(\alpha - \delta_t)dt + \sigma_\delta d\mathbf{W}_\delta, \tag{3}$$

$$d\boldsymbol{\gamma}_t = -k_\gamma \gamma_t dt + \sigma_\gamma d\mathbf{W}_\gamma, \tag{4}$$

$$\begin{aligned} &Ed\mathbf{W}_G d\mathbf{W}_\delta = \rho_{\delta G}dt, \quad Ed\mathbf{W}_\delta d\mathbf{W}_\gamma = \rho_{\delta\gamma}dt, \\ &Ed\mathbf{W}_G d\mathbf{W}_\gamma = \rho_{G\gamma}dt. \end{aligned} \tag{5}$$

Here, at time t, $\mathbf{G}_t := \ln(\mathbf{S}_t)$ is the logarithm of the spot price, $\boldsymbol{\delta}_t + \boldsymbol{\gamma}_t$ is the spot (instantaneous) convenience yield with the medium-term $\boldsymbol{\delta}_t$ and the short-term $\boldsymbol{\gamma}_t$ mean-reverting factors having long-run means α and 0 respectively in the market measure, λ_G is the *market price of risk* premium of the G process and $\mathbf{W}_{G,}$ $\mathbf{W}_\delta$ and $\mathbf{W}_\gamma$ are Wiener processes with $\sigma_{G,}$ σ_δ and σ_γ their corresponding volatilities.

In the *risk-neutral measure* this system becomes

$$d\mathbf{G}_t = \left(r^f - \delta_t - \gamma_t - \frac{1}{2}\sigma_G^2\right)dt + \sigma_G d\mathbf{W}_\mathbf{G}^\mathbf{Q}, \tag{6}$$

$$d\boldsymbol{\delta}_t = k_\delta(\alpha - \delta_t - \lambda_\delta)dt + \sigma_\delta d\mathbf{W}_{\boldsymbol{\delta}}^\mathbf{Q}, \tag{7}$$

$$d\boldsymbol{\gamma}_t = k_\gamma(-\gamma_t - \lambda_\gamma)dt + \sigma_\gamma d\mathbf{W}_{\boldsymbol{\gamma}}^\mathbf{Q}, \tag{8}$$

$$\begin{aligned} &Ed\mathbf{W}_\mathbf{G}^\mathbf{Q} d\mathbf{W}_{\boldsymbol{\delta}}^\mathbf{Q} = \rho_{\delta G}dt, \quad Ed\mathbf{W}_{\boldsymbol{\delta}}^\mathbf{Q} d\mathbf{W}_{\boldsymbol{\gamma}}^\mathbf{Q} = \rho_{\delta\gamma}dt, \\ &Ed\mathbf{W}_\mathbf{G}^\mathbf{Q} d\mathbf{W}_{\boldsymbol{\gamma}}^\mathbf{Q} = \rho_{G\gamma}dt, \end{aligned} \tag{9}$$

where $k_\delta\lambda_\delta$ and $k_\gamma\lambda_\gamma$ are, respectively, the market risk premia for the $\boldsymbol{\delta}$ and $\boldsymbol{\gamma}$ processes.

Setting the $\boldsymbol{\gamma}$ factor identically equal to zero, (2) to (9) becomes the Gibson–Schwartz (1990) model so that our model is its extension, but with convenience yield decomposed into two parts, $\boldsymbol{\delta}$ and $\boldsymbol{\gamma}$, with different mean-reversion speeds.

Defining $x_t := (1/k_\delta)(\boldsymbol{\delta}_t - \alpha)$, $\mathbf{y}_t := \boldsymbol{\gamma}_t/k_\gamma$ and $\mathbf{p}_t := \mathbf{G_t} - \mathbf{x}_t - \mathbf{y}_t$ in the market measure, we have

$$d\mathbf{x}_t = \frac{1}{k_\gamma}d\boldsymbol{\delta}_t = -k_\delta x_t dt + \frac{\sigma_\delta}{k_\delta}d\mathbf{W}_\delta, \tag{10}$$

$$d\mathbf{y}_t = \frac{1}{k_\gamma}d\boldsymbol{\gamma}_t = -k_\gamma y_t dt + \frac{\sigma_\gamma}{k_\gamma}d\mathbf{W}_\gamma, \tag{11}$$

† Boldface is used throughout to denote random entities, here conditional.

$$d\mathbf{p}_t = d\mathbf{G}_t - \frac{d\boldsymbol{\delta}_t}{k_\delta} - \frac{d\boldsymbol{\gamma}_t}{k_\gamma} = \left(r^f + \lambda_G - \alpha - \frac{1}{2}\sigma_G^2\right)dt + \sigma_G d\mathbf{W}_G - \frac{\sigma_\delta}{k_\delta}d\mathbf{W}_\delta - \frac{\sigma_\gamma}{k_\gamma}d\mathbf{W}_\gamma. \tag{12}$$

Setting $k_x := k_\delta$, $k_y := k_\gamma$, $\sigma_x := \sigma_\delta/k_\delta$, $\sigma_y := \sigma_\gamma/k_\gamma$, $\lambda_x := \lambda_\delta/k_\delta$, $\lambda_y := \lambda_\gamma/k_\gamma$, $\lambda_p := \lambda_G - \lambda_\delta - \lambda_\gamma$, $u := r^f + \lambda_G - \alpha - (1/2)\sigma_G^2$, $\sigma_p^2 := \sigma_G^2 + \sigma_x^2 + \sigma_y^2 + 2\rho_{\delta\gamma}\sigma_x\sigma_y - 2\rho_{G\delta}\sigma_G\sigma_x - 2\rho_{G\gamma}\sigma_G\sigma_y$ and $d\mathbf{W}_x := d\mathbf{W}_\delta$, $d\mathbf{W}_y := d\mathbf{W}_\gamma$, $d\mathbf{W}_p := (1/\sigma_p)(\sigma_G d\mathbf{W}_G - (\sigma_\delta/k_\delta)d\mathbf{W}_\delta - (\sigma_\gamma/k_\gamma)d\mathbf{W}_\gamma)$, the original model in the market measure becomes

$$\ln(\mathbf{S}_t) = \mathbf{x}_t + \mathbf{y}_t + \mathbf{p}_t, \tag{13}$$

$$d\mathbf{x}_t = -k_x x_t dt + \sigma_x d\mathbf{W}_\mathbf{x}, \tag{14}$$

$$d\mathbf{y}_t = -k_y y_t dt + \sigma_x d\mathbf{W}_\mathbf{y}, \tag{15}$$

$$d\mathbf{p}_t = u dt + \sigma_p d\mathbf{W}_\mathbf{p}, \tag{16}$$

$$Ed\mathbf{W}_x d\mathbf{W}_y = \rho_{xy}dt, \quad Ed\mathbf{W}_x d\mathbf{W}_p = \rho_{xp}dt, \quad Ed\mathbf{W}_y d\mathbf{W}_p = \rho_{yp}dt, \tag{17}$$

where $\mathbf{x}$ is the *short-term* factor with mean-reversion speed k_x and volatility $\sigma_{x,}$ $\mathbf{y}$ is the *medium-term* factor with mean-reversion speed k_y and volatility $\sigma_{y,}$ $\mathbf{p}$ is the *long-term trend* factor with *growth rate u* and volatility $\sigma_{p,}$ and $\boldsymbol{W}_{x,}$ $\boldsymbol{W}_y$ and $\boldsymbol{W}_p$ are all Wiener processes. Note that factors $\mathbf{x}$ and $\mathbf{y}$ both have *zero* long-run means so that they will fluctuate around the trend factor $\mathbf{p}$.

In the risk-neutral measure the above system becomes

$$\ln(\mathbf{S}_t) = \mathbf{x}_t + \mathbf{y}_t + \mathbf{p}_t, \tag{18}$$

$$d\mathbf{x}_t = k_x(-x_t - \lambda_x)dt + \sigma_x d\mathbf{W}_\mathbf{x}^\mathbf{Q}, \tag{19}$$

$$d\mathbf{y}_t = k_y(-y_t - \lambda_y)dt + \sigma_y d\mathbf{W}_\mathbf{y}^\mathbf{Q}, \tag{20}$$

$$d\mathbf{p}_t = (u - \lambda_p)dt + \sigma_p d\mathbf{W}_\mathbf{p}^\mathbf{Q} \tag{21}$$

$$Ed\mathbf{W}_\mathbf{x}^\mathbf{Q} d\mathbf{W}_\mathbf{y}^\mathbf{Q} = \rho_{xy}dt, \quad Ed\mathbf{W}_\mathbf{x}^\mathbf{Q} d\mathbf{W}_\mathbf{p}^\mathbf{Q} = \rho_{xp}dt, \quad Ed\mathbf{W}_\mathbf{y}^\mathbf{Q} d\mathbf{W}_\mathbf{p}^\mathbf{Q} = \rho_{yp}dt, \tag{22}$$

where $k_x\lambda_{x,}$ $k_y\lambda_y$ and λ_p are the *risk premia* of factors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{p}$ respectively.†

We term the model (13) to (22) the *three factor (log) spot price* model. It determines spot price fluctuations in terms of three components: two mean-reverting factors representing short- and medium-term economic forces and one long-term factor that reflects the equilibrium commodity price trend and captures permanent price shocks. We note that the model belongs to the exponential affine class in the framework of Duffie *et al.* (2000). ‡

Solving (19), (20) and (21), and substituting into (18), together with taking logarithms of the no-arbitrage condition for the price at t of the futures contract with maturity T given by

$$F(t, T) = E_t^Q[\mathbf{S}_T], \tag{23}$$

in terms of the conditional expectation in the risk-neutral measure Q at t, yields $\ln F(t, T)$ in terms of the three factors at t as

$$\begin{aligned}\ln F(t, T) = {} & (x_t + \lambda_x)e^{-k_x(T-t)} + (y_t + \lambda_y)e^{-k_y(T-t)} \\ & + p_t - (\lambda_x + \lambda_y) + (u - \lambda_p)(T - t) \\ & + \frac{1}{2}\left[\begin{aligned} & \frac{1 - e^{-2k_x(T-t)}}{2k_x}\sigma_x^2 \\ & + \frac{1 - e^{-2k_y(T-t)}}{2k_y}\sigma_y^2 + \sigma_p^2(T - t) \\ & + \frac{2(1 - e^{-(k_y+k_x)(T-t)})}{k_x + k_y}\rho_{xy}\sigma_x\sigma_y \\ & + \frac{2(1 - e^{-k_x(T-t)})}{k_x}\rho_{xp}\sigma_x\sigma_p \\ & + \frac{2(1 - e^{-k_y(T-t)})}{k_y}\rho_{yp}\sigma_y\sigma_p \end{aligned}\right].\end{aligned} \tag{24}$$

3.2. Two-factor convenience yield

The price at t of the futures contract with maturity T is given in terms of the instantaneous convenience yield at t by

$$F(t, T) = S_t \exp\left(r^f(T - t) - \int_t^T \delta(t, s)ds\right), \tag{25}$$

where $\boldsymbol{\delta}(t, s)$ is the *instantaneous convenience yield* at time t of the contract with maturity s. In our model

$$\begin{aligned}\delta(t, T) = {} & r^f + (\mathbf{x}_t + \lambda_x)k_x e^{-k_x(T-t)} + (\mathbf{y}_t + \lambda_y)k_y e^{-k_y(T-t)} \\ & - (u - \lambda_p) - \frac{1}{2}[e^{-2k_x(T-t)}\sigma_x^2 + e^{-2k_y(T-t)}\sigma_y^2 + \sigma_p^2 \\ & + 2\rho_{xy}\sigma_x\sigma_y e^{-(k_x+k_y)(T-t)} + 2\rho_{xp}\sigma_x\sigma_p e^{-k_x(T-t)} \\ & + 2\rho_{yp}\sigma_y\sigma_p e^{-k_y(T-t)}].\end{aligned} \tag{26}$$

When $T \to t$ this reduces to the (*instantaneous*) *spot convenience yield* given by

$$\begin{aligned}\boldsymbol{\delta}_t = \boldsymbol{\delta}(t, t) = {} & r^f + k_x(\mathbf{x}_t + \lambda_x) + k_y(\mathbf{y}_t + \lambda_y) - u + \lambda_p \\ & - \frac{1}{2}(\sigma_x^2 + \sigma_y^2 + \sigma_p^2 + 2\rho_{xy}\sigma_x\sigma_y \\ & + 2\rho_{xp}\sigma_x\sigma_p + 2\rho_{yp}\sigma_y\sigma_p),\end{aligned} \tag{27}$$

so that the spot convenience yield $\boldsymbol{\delta}$ is seen to be an affine combination of $\mathbf{x}$ and $\mathbf{y}$ factors as designed. Our calibration of the three-factor model for oil futures shows, as expected, that the $\mathbf{x}$ factor has a much higher mean-reversion speed than that of the $\mathbf{y}$ factor (see table 3). As a consequence, (25) and (26) imply that

† Note that, since these risk premia for the $\mathbf{x}$ and $\mathbf{y}$ factors are *constant*, the mean-reversion speeds of these factors are the same under the market and risk-neutral measures (see Casassus and Collin-Dufresne (2005) who assume risk premia stochastic).
‡ See last footnote section 1.

the longer-term convenience yields $\boldsymbol{\delta}(t, T)$ are determined mainly by the **y** factor, while the spot convenience yield $\boldsymbol{\delta}_t$ is determined mainly by the **x** factor.

Although convenience yield is a concept of the theory of storage, in the context of the theory of normal backwardation we wish to know the shape and overall slope (i.e. *contango,* upwards slope, or *backwardation,* downwards slope) of the futures price term structure. Taking logarithms of both sides of (25) and differentiating the result with respect to maturity gives

$$\frac{\partial(F(t,T))/\partial T}{F(t,T)} = r^f - \delta(t,T). \tag{28}$$

The convenience yield $\delta(t, T)$ therefore determines the sign of the slope of the term structure of futures prices and the **x** and **y** factors can be regarded as two components of this slope. Thus, when the instantaneous convenience yield $\boldsymbol{\delta}(t, T)$ is strictly less than the instantaneous risk-free rate r^f, futures prices are in contango *locally* in maturity. However, because **x** and **y** may have different signs at a specific time *t,* the three-factor model is capable of reproducing the empirical near term U or humped futures curves of figure 1.

3.3. *Results*

The appendix shows the state space form of the three-factor model needed for the parameter estimation filtering technique.† Since the three factors of our model are not directly observable, to calibrate it we use the *EM algorithm* procedure which alternates between the *Kalman filter* and maximum likelihood parameter estimation of the model in state space form to convergence (Schwartz 1997, Schwartz and Smith 2000, Geman and Nguyen 2005).

By F1, F3, F6, F9, F12, F15, and F17 we denote respectively the 1st, 3rd, 6th, 9th, 12th, 15th and 17th month futures contracts (in the order of their maturities) which we use in the calibration of the three-factor spot price model.‡ Table 3 shows the parameter estimates. These estimated parameters are nearly all significant except for the risk premia. Both the **x** and **y** factors are significantly mean-reverting, as can be seen in figure 4, which shows the estimated paths of the three factors. From the estimated parameters the short-term factor **x** has a half-life of about 2.5 months with a volatility of 20%, the medium-term factor **y** has a half-life of about 9.5 months with volatility 28% and the long-term factor **p** has a volatility of about 20%.

Table 3. Parameter estimates of two- and three-factor models.

Variable	Three-factor model	Two-factor model
k_x	3.4152 (0.1147)	
k_y	0.8802 (0.0384)	1.0715 (0.0166)
U	0.0809 (0.0425)	0.0838 (0.0428)
σ_x	0.1977 (0.0078)	
σ_y	0.2817 (0.0078)	0.2866 (0.0067)
σ_p	0.1953 (0.0051)	0.1960 (0.0044)
λ_x	−0.0205 (0.0128)	
λ_y	0.1600 (0.0701)	0.0720 (0.0581)
λ_p	0.0731 (0.0425)	0.1063 (0.0428)
ρ_{xy}	−0.0794 (0.0518)	
ρ_{xp}	0.0838 (0.0431)	
ρ_{yp}	−0.0067 (0.0472)	0.0932 (0.0346)
ξ_1	0.0160 (0.0037)	0.0347 (0.0072)
ξ_2	0.0004 (0.0000)	0.0115 (0.0025)
ξ_3	0.0014 (0.0006)	0.0000 (0.0000)
ξ_4	0.0016 (0.0005)	0.0022 (0.0006)
ξ_5	0.0011 (0.0007)	0.0021 (0.0009)
ξ_6	0.0158 (0.0033)	0.0161 (0.0034)
ξ_7	0.0160 (0.0035)	0.0172 (0.0036)
Log-likelihood	**24 493**	22331

Note: $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7$ are, respectively, the pricing errors of the F1, F3, F6, F9, F12, F15 and F17 contracts. The quantities in parentheses are (asymptotic) standard deviations.

3.4. *Comparison with two factor models*

To see whether or not the three-factor model is significantly better than two-factor models, we remove the **x** factor and re-estimate the resulting model with only the **y** and **p** factors.§ Table 3 demonstrates that the pricing errors $(\xi_1, \ldots, \xi_7)$ are generally smaller than those reported for the Gibson–Schwartz model by Schwartz (1997). Comparing our two- and three-factor models, the inclusion of the **x** factor significantly improves the data fit, according to the likelihood ratio test,¶ and reduces pricing errors for short-term contracts.

4. Model interpretation

In this section we study the relations between the historical paths of various economic variables and those of our latent factors (figure 4), estimated in terms of the means of the sequential posterior Gaussian state distributions obtained from the Kalman filter using the optimal parameter estimates of the final iteration of the EM algorithm. Since commodities are both real and

† Note that in order to make a comparison with the results of the two-factor models of Gibson and Schwartz (1990) and Schwartz and Smith (2000), we employ the same estimation method used in those papers. We also used the method proposed by Dempster and Tang (2011) to eliminate mean-reversion parameter estimation errors, but the results reported here are little changed.
‡ We did not include the longer futures for three reasons: (1) For the open interest shown in figure 2, we see that the liquidity of longer futures contracts decreases quickly. As a result, the prices of contracts with maturities longer than 17 months may not be able to reflect 'true' market information. (2) Moreover, longer-term futures prices for oil in the early period of our dataset do not exist. (3) The novelty in this paper is the introduction of the short-term **x** factor. We thus followed the standard Schwartz–Smith methodology in estimating the medium- and long-term factors and used similar futures times to maturity.
§ We have seen in section 3.1 that this two-factor model is the same as the Gibson–Schwartz (1990) model.
¶ Note that this test applies to our nested case. The likelihood ratio test statistic is 4324, highly significant at the 99% confidence level for the Chi squared distribution with five degrees of freedom.

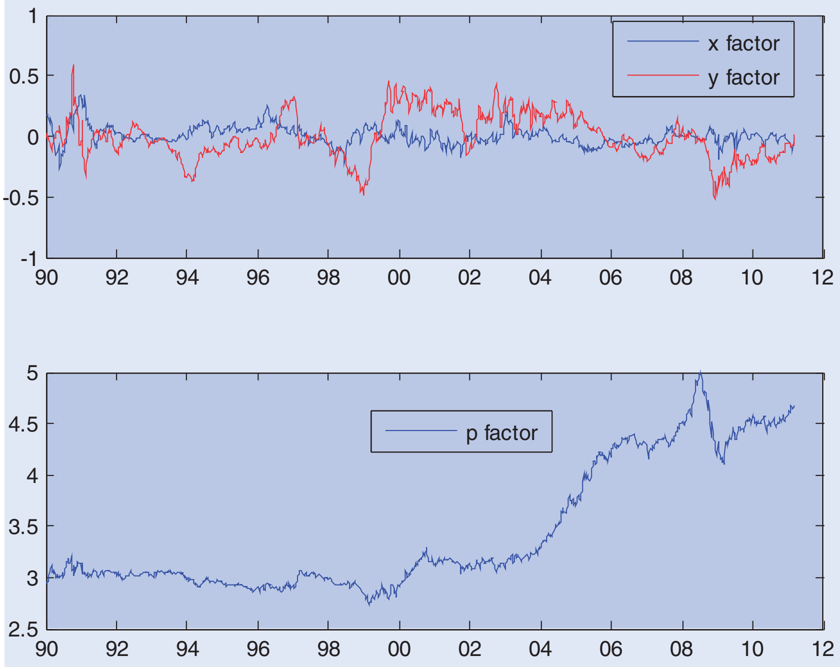

Figure 4. Estimated latent factor evolution in the three-factor model.

financial assets, we expect that both fundamental and financial variables will play an important role in explaining the three factors.

4.1. Explanatory variable specification

We classify our explanatory variables into four categories: (1) variables from other financial markets, such as US dollar index returns, SP500 equity index returns, etc.; (2) variables indicating the phase of the business cycle, such as the coincident business cycle index and the term spread on US interest rates; (3) variables indicating net demand for oil, such as the oil inventory level and the heating oil–crude oil spread; (4) trading variables, such as growth rate of open interest and hedging pressure for oil futures contracts.

More specifically, we utilized the following variables at weekly frequency.

4.1.1. Financial variables.

- **USD (y1)**: Weekly returns of the US dollar index. This index measures the performance of the US dollar against a basket of currencies. It goes up when the US dollar gains strength relative to other currencies.
- **SP500 (y2)**: Weekly returns of the S&P500 equity index. Inclusion of SP500 returns controls for the possibility that investors were pursuing trading strategies in oil futures that are conditional on equity markets.
- **LIBOR (y3)**: The weekly level of the three month US LIBOR rate. Casassus and Colin-Dufresne (2005) and Frankel (2008) show that interest rates tend to influence the willingness to store inventory and thus will influence the convenience yield.†
- **VIX (y4)**: The weekly level of the VIX index for the equity market. This index represents one measure of the market's expectation of stock market volatility over the next 30-day period. It is a weighted blend of prices for a range of options on the S&P 500 index.
- **CreditSpread (y5)**: The weekly spread between Moody's BAA-rated and AAA-rated corporate bond yields, following Bailey and Chan (1993), as a proxy for the default premium.

4.1.2. Business cycle variables.

- **TermSpread (y6)**: The weekly spread between the 10 year and three month Treasury bond yields. The term spread between the long-term and short-term interest rates has often been found to be the most important predictor of economic recessions (see, e.g., Estrella and Mishkin (1998)).
- **CoinIndex (y7)**: The *change* in weekly levels of the business cycle coincident index, as this

† For example, a higher interest rate corresponds to a higher marginal cost of storage, a higher convenience yield and a futures term structure more likely to be in backwardation, as Keynes (1930) described. The current low interest rates have led through the opposite effect to an oil futures market in contango and a frenzy of oil storage building to exploit it physically (Bouchouev 2011).

index, obtained from the Conference Board Inc., is not stationary. However, since the coincident index is calculated at monthly frequency, we assume the weekly values of coincident index change are identical in each month.

4.1.3. Demand variables.

- **HOCLSpread (y8)**: The weekly *log* spread between heating and crude oil prices. Since heating oil is the main product of crude oil, the (log) price spread can reflect the relative scarcity of crude oil (Casassus *et al*. 2010). Note that a high spread means that crude oil is cheap relative to heating oil and hence has low convenience yield, a phenomenon independent of heating oil, and hence spread, seasonality.
- **Inventory (y9)**: US weekly crude oil inventory (excluding strategic petroleum reserves) in millions of barrels obtained from the US Energy Information Administration. Since this data is non-stationary, we follow Gorton *et al*. (2007) in first applying a Hodrick–Prescott (HP) filter to the whole time series. The detrended stationary part is used in our empirical analysis.

4.1.4. Trading variables.

- **OpenInterest (y10)**: The weekly growth rate (log difference) of open interest for all oil contracts, obtained from the US Commodity Futures Trading Commission (CFTC).
- **HedgingPressure (y11)**: The weekly hedging pressure, obtained from the US Commodity Futures Trading Commission (CFTC). This measure is calculated using the hedgers' short position less their long position normalized by total open interest.

4.2. *SVAR model statement*

We estimate a *structural vector auto-regression* (SVAR) model (Sims 1980, Hamilton 1994) to address the relationship between the three latent factors and the 11 explanatory variables, namely

$$BY_t = AY_{t-1} + \Sigma\varepsilon_{Y,t}, \tag{29}$$

where the vector of the variables is given by $Y_t := (y_{1,t}, \ldots, y_{11,t}, z_t')'$, with z_t a 3 vector representing the latent **x, y** and **p** factors, A and B are 14×14 matrices and $\varepsilon_{Y,t}$ is a vector of Gaussian disturbances with a spherical covariance matrix.

In structuring the SVAR matrix B we assume that all variables can influence the latent **x, y** or **p** factors, but that, consistent with our three-factor model, these factors do not (directly) influence each other. Similarly, the financial and business cycle variables do not directly influence each other, but they do affect the fundamental and trading variables. On the other hand, these latter variables do *not* influence financial and business cycle variables, i.e. the two blocks of variables are in cascade (causal Wold recursive) form. We also assume that the heating oil–crude oil spread can influence oil inventory and *vice versa,* i.e. they are determined simultaneously. Further, we assume that the fundamental demand and supply variables can influence futures trading and *vice versa,* but that open interest and hedgers' positions do not influence each other. These assumptions lead to a B matrix of (29), corresponding to each of the three latent factors *separately* in turn, in lower triangular form given by

$$B = \begin{bmatrix} 1 & & & & & & & & & & & \\ 0 & 1 & & & & & & & & & & \\ 0 & 0 & 1 & & & & & & & & & \\ 0 & 0 & 0 & 1 & & & & & 0 & & & \\ 0 & 0 & 0 & 0 & 1 & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 1 & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & & & \\ x & x & x & x & x & x & x & 1 & & & & \\ x & x & x & x & x & x & x & x & 1 & & & \\ x & x & x & x & x & x & x & x & x & 1 & & \\ x & x & x & x & x & x & x & x & x & 0 & 1 & \\ x & x & x & x & x & x & x & x & x & x & x & 1 \end{bmatrix},$$

with x representing non-zero elements. Note that, by using this B matrix, the full model can be estimated under our assumptions by separately estimating the resulting model for each latent factor in turn.

4.3. *Results*

To see how the exogenous variables influence our three latent variables, we first analyse the impulse response functions of the estimated SVAR model. Figures 5, 6 and 7 show, for each of the **x, y** and **p** factors respectively, impulse response functions for a one standard deviation positive shock to each of the exogenous explanatory variables in turn. Note that the vertical scales in these diagrams are variable. The gray area in each represents the 95% confidence level obtained from bootstrapping.

4.3.1. x factor impulse responses. The LIBOR rate impacts positively on the short-term convenience yield **x** factor, which is consistent with the standard argument of the theory of storage, i.e. a higher interest rate will result in a higher marginal cost of storing commodities. Thus a higher LIBOR rate should correspond to a higher convenience yield. We also see that credit spread co-moves with the **x** factor, which is consistent with Acharya *et al.* (2008) in that a higher default risk tends to encourage more commodity producers' hedging and thus a futures curve in deeper backwardation.

The log spread between heating and crude oil influences the **x** factor negatively. A high spread means a relatively low price of crude oil and less demand for it (or more crude oil inventory), corresponding to a lower convenience yield. For details of the equilibrium relationship between oil convenience yield and the heating oil–crude oil spread, see Casassus *et al.* (2010).

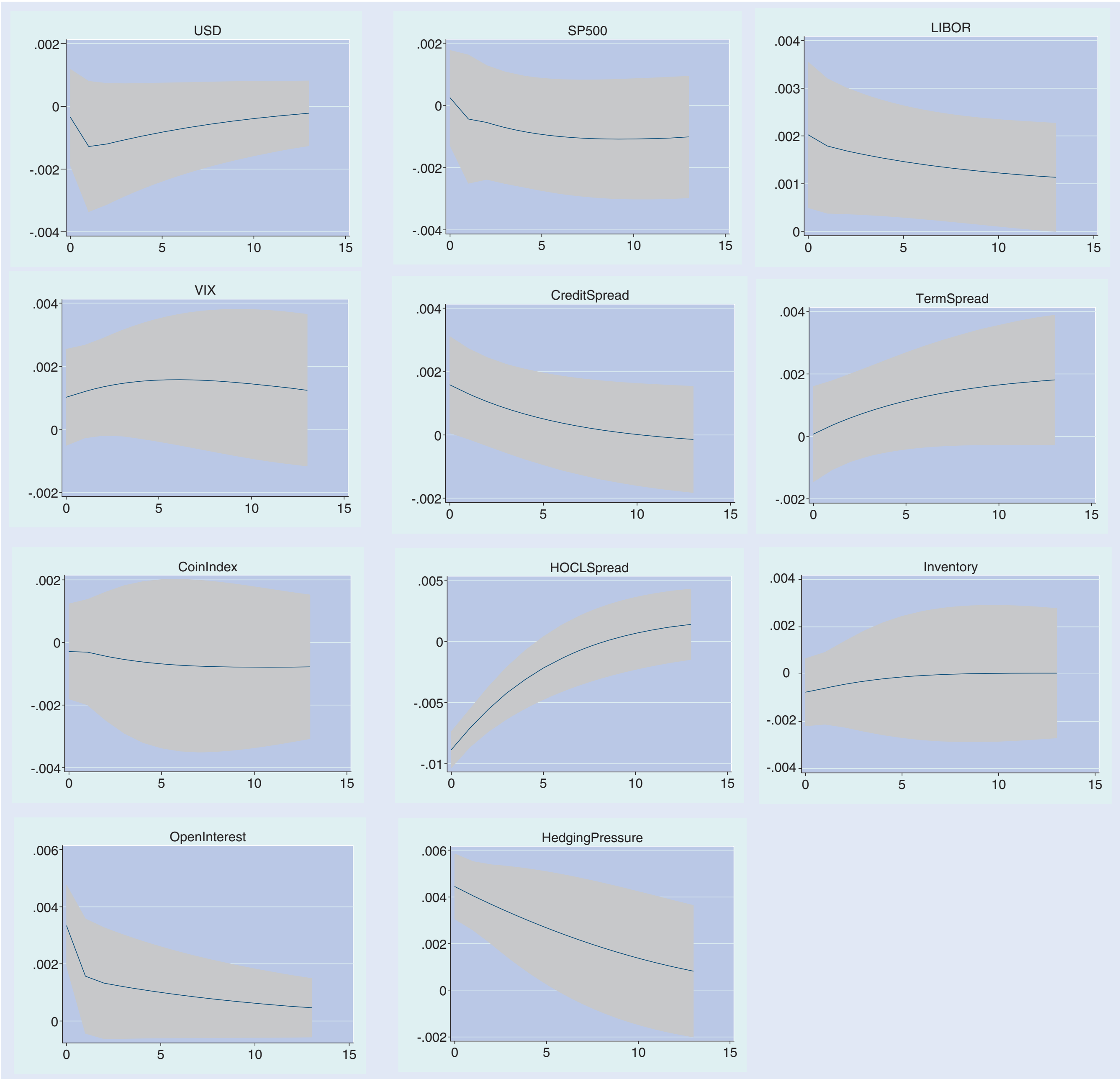

Figure 5. Impulse responses of the **x** factor to a one standard deviation positive shock of each of the 11 explanatory variables.

As shown by Hong and Yogo (2010), open interest in commodity futures forecasts commodity returns. Our result shows that a higher open interest growth rate corresponds to a futures term structure in deeper backwardation. We also see that more hedging pressure corresponds to deeper backwardation. This is consistent with Keynes (1930) and Hirshleifer (1990), i.e. the more futures contracts sold the deeper the backwardation of the futures term structure.

4.3.2. y factor impulse responses. Similar to the situation with the **x** factor, we see that the LIBOR rate also impacts positively on the medium-term convenience yield **y** factor, which is again consistent with the theory of storage. We also see that the VIX correlates positively with the **y** factor. Since the VIX index can be seen as an indicator of the volatility of financial markets, the theory of storage says that higher volatility will lead to larger convenience yields, resulting here in a positive relationship between the VIX and the **y** factor.

Both the term spread and the business cycle coincident index have a positive impact on the **y** factor. This is because a high term spread and coincident index both correspond to a booming state of the economy with a higher oil demand and hence a higher convenience yield. The log spread between heating and crude oil also has a negative impact on the **y** factor; the explanation for this is

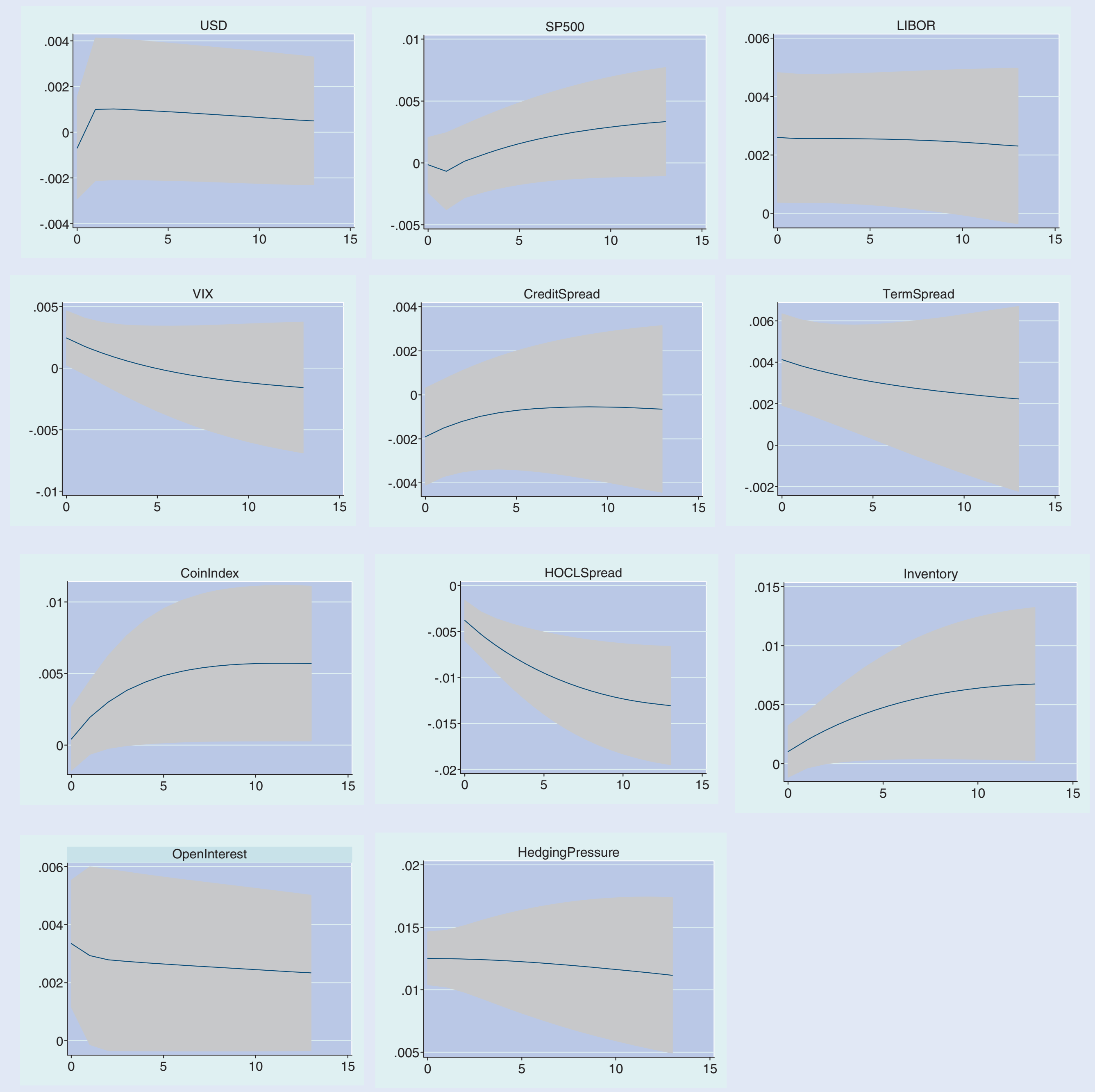

Figure 6. Impulse responses of the **y** factor to a one standard deviation positive shock of each of the 11 explanatory variables.

similar to that for the **x** factor. The inventory has a positive impact on the **y** factor, which is consistent with the theory of storage, i.e. high inventory results in higher convenience yield and deeper backwardation of the futures term structure (see (28)). Similar to the **x** factor, the **y** factor is also affected by the growth of open interest and the hedging pressure, but at lower impact levels.

4.3.3. p factor impulse responses. First observe from figure 7 that the impulse responses of the long-term **p** factor to explanatory variable shocks are of larger magnitude and converge faster to equilibrium than those of the **x** and **y** factors, as is consistent with its GBM dynamics.

The US dollar index co-moves negatively with the permanent shock **p** factor. This is because oil is traded in dollars, hence the depreciation of dollars should increase the price of oil due to the numeraire effect. This effect tends to be 'permanent', i.e. only affecting the long-term **p** factor in our model.

There have been several studies of the relationship between the stock market portfolio and futures prices (e.g., Dusak (1973) and Holthausen and Hughes (1978)). Using *t*-tests they found that no correlation exists between futures and market portfolio returns.

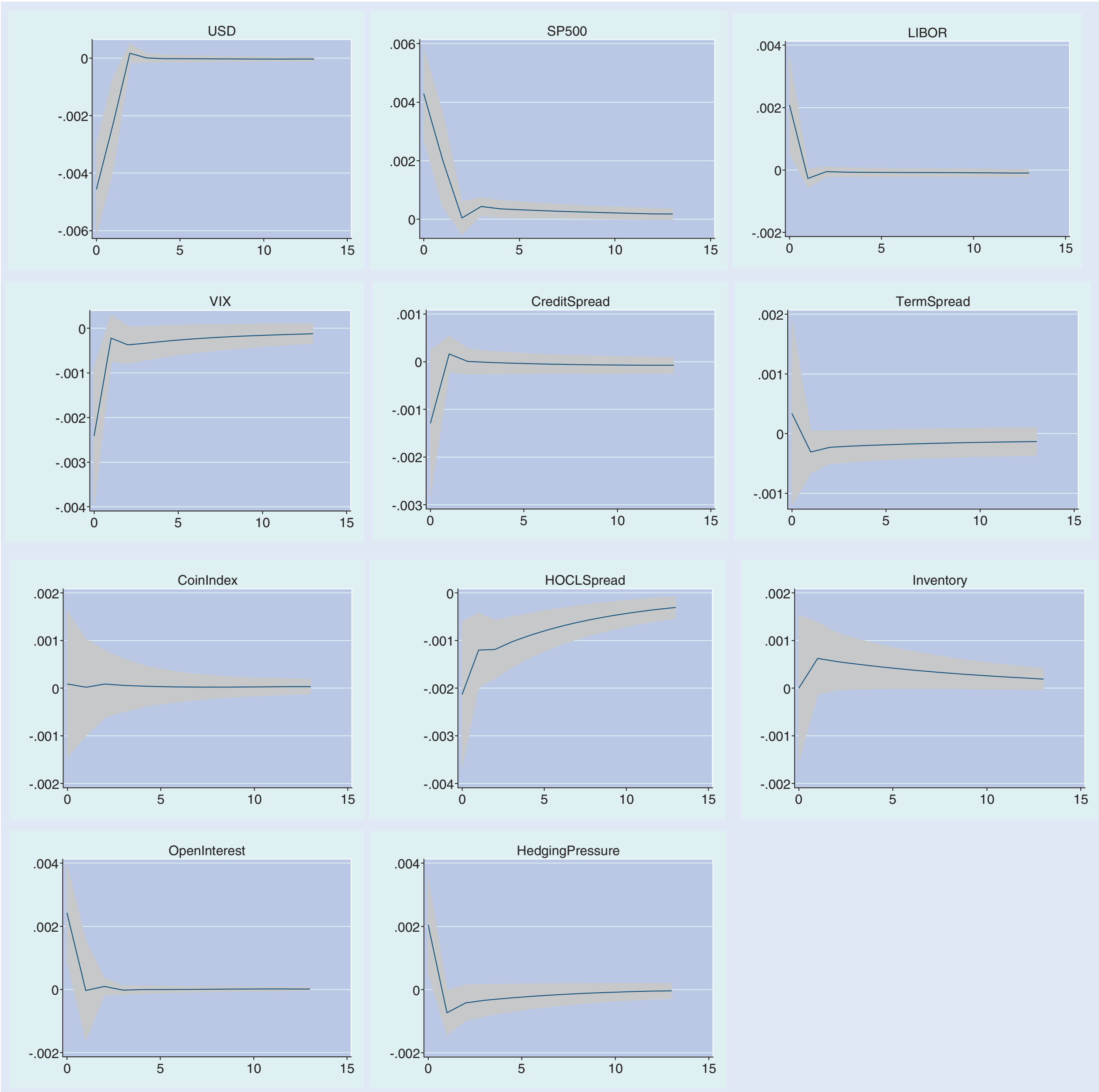

Figure 7. Impulse responses of the **p** factor to a one standard deviation positive shock of each of the 11 explanatory variables.

However, by decomposing oil futures prices into three factors, we see that the long-term **p** factor does co-move with SP500 index returns, but the **x** and **y** factors do not.

Similar to the **x** and **y** factors, the heating oil–crude oil spread has a negative impact on the **p** factor as well.

4.3.4. Forecast error variance decomposition. We complement the conclusions derived from impulse response analysis of our estimated SVAR model with the forecast error variance decomposition from one- and 13-step ahead rolling forecasts (see, e.g., Lütkepohl (2007)). Table 4 shows the variance decomposition for both one week and one quarter forecasts, with entries that are percentages of the forecast error variance of each factor accounted for by exogenous shocks to each explanatory variable, or an average of these percentages for a group of variables.

We see that the variability of the log spread between heating and crude oil strongly influences that of the **x** factor; however, this effect decreases rapidly as the forecasting horizon lengthens. Trading variables such as the growth of the open interest and the hedging pressure also play an important role. The **y** factor is influenced by the variability of the business cycle variables, term spread

Table 4. Forecast error variance decomposition.

	x factor		**y** factor		**p** factor	
Variable	1-week horizon	13-week horizon	1-week horizon	13-week horizon	1-week horizon	13-week horizon
USD	0.02	0.2	0.03	0.06	**2.93**	**3.51**
SP500	0.00	0.25	0.00	0.35	**2.60**	**3.14**
LIBOR	0.62	0.70	0.48	0.53	0.61	0.60
VIX	0.15	0.65	0.43	0.12	0.82	0.88
Credit spread	0.38	0.17	0.26	0.08	0.23	0.23
Term spread	0.00	0.50	**1.21**	0.80	0.02	0.07
Coin index	0.01	0.14	0.01	**1.85**	0.00	0.00
HO–-CO spread	**11.88**	**4.84**	**1.03**	**8.31**	0.64	1.59
Inventory	0.89	0.03	0.08	**2.11**	0.00	0.27
Open interest	**1.68**	0.55	0.80	0.60	0.83	0.79
Hedging pressure	**3.00**	**2.38**	**11.16**	**12.24**	0.59	0.70
Aggregate averages						
Financial	0.23	0.39	0.24	0.23	**1.44**	**1.67**
Business cycle	0.01	0.32	0.61	**1.33**	0.01	0.04
Fundamental	**6.39**	**2.44**	0.56	**5.21**	0.32	0.93
Trading	**2.34**	**1.47**	**5.98**	**6.42**	0.71	0.75

Note: This table reports the forecasting error variance decomposition for each factor over two rolling forecast horizons—one week and one quarter. Quantities in the table are percentages of the total factor variance.

and coincident index. The fundamental demand and supply variables, heating-crude oil spread and the inventory also play a role, but their role is only significant at the 13-week forecasting horizon. Futures hedging pressure has a very strong influence on the variability of the medium-term **y** factor, but the long-term **p** factor is mainly influenced by the US dollar index and SP500 returns.

4.3.5. Summary. The financial variables mainly affect the **p** factor, however they also have a minimal influence on the **x** and **y** factors. The business cycle variables influence mainly the **y** factor; a booming state corresponding to larger **y** factors (deeper long-end backwardation). The fundamental supply and demand variables affect both **x** and **y** factors; higher net demand results in higher **x** factor (deeper short-end backwardation) and **y** factor (deeper long-end backwardation) levels. The trading variables influence both **x** and **y** factors; more intensive futures trading and stronger hedging pressure result in higher **x** and **y** factors.

5. Conclusion

In this paper we find that the two-factor models in the literature are not able to model the whole crude oil futures price term structure, especially at the short end. Hence we propose a three-factor model for commodity futures prices. This model is shown to be an extension of the Gibson–Schwartz (1990) (Schwartz–Smith 2000) model. An affine combination of the **x** and **y** factors in our model represents convenience yield, while the third **p** factor models long-term trend. By regressing the three factors on several economic variables using an SVAR model, we see that the short-term **x** factor is highly correlated with demand and trading variables. The medium-term **y** factor has a relationship with the business cycle, net oil demand and trading variables. The long-term **p** factor is mainly related to financial variables. The business cycle and fundamental variables affect the movement and the shape of the oil futures price term structure; but financial and trading variables do as well. This phenomenon reflects the fact that commodities combine the characteristics of both real and financial assets.

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Appendix A: Three-factor model in state space form

The state space form of a dynamic statistical model consists of a transition and a measurement equation. The *transition equation* describes the dynamics of the data-generating process of unobservable *state variables*. In our model this is a discrete-time version of (14) to (16). The *measurement equation* relates a multivariate time series of *observable variables,* here the future prices of different maturities, to the unobservable vector of state variables, the **x, y** and **p** factors. The measurement equation is obtained from (24) by adding *uncorrelated* noise to take into account *pricing errors*.† These errors may be caused by bid–ask spreads, non-simultaneity of the observations, etc.

In more detail, suppose the data are sampled at equally spaced times $t_{n,}\ n=1,\ldots,N$, and that $\Delta:=t_{n+1}-t_{n,}$ is the interval between two observations. Let $X_n := [x_{t_n}\ y_{t_n}\ p_{t_n}]'$ represent the vector of state variables at time t_n where the prime denotes transpose. Discretizing (14) to (16) we obtain the *transition equation* as

$$\mathbf{X}_{n+1} = AX_n + b + \mathbf{w}, \tag{A.1}$$

where **w** is a Gaussian random noise vector with mean 0 and covariance matrix Q and A, b and Q are given by

$$A = \begin{bmatrix} e^{-k_x\Delta} & 0 & 0 \\ 0 & e^{-k_y\Delta} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{A.2}$$

† But see Dempster and Tang (2011) for more general assumptions.

$$b = \begin{bmatrix} 0 & 0 & u\Delta \end{bmatrix}', \tag{A.3}$$

$$Q = \begin{bmatrix} \sigma_x^2 \frac{1-e^{(-2k_x\Delta)}}{2k_x} & \frac{\rho_{xy}\sigma_x\sigma_y}{k_x+k_y}(1-e^{-(k_x+k_y)\Delta}) & \frac{\rho_{xp}\sigma_x\sigma_p}{k_x}(1-e^{-k_x\Delta}) \\ \frac{\rho_{xy}\sigma_x\sigma_y}{k_x+k_y}(1-e^{-(k_x+k_y)\Delta}) & \sigma_y^2 \frac{1-e^{(-2k_y\Delta)}}{2k_y} & \frac{\rho_{yp}\sigma_y\sigma_p}{k_y}(1-e^{-k_y\Delta}) \\ \frac{\rho_{xp}\sigma_x\sigma_p}{k_x}(1-e^{-k_x\Delta}) & \frac{\rho_{yp}\sigma_y\sigma_p}{k_y}(1-e^{-k_y\Delta}) & \sigma_p^2\Delta \end{bmatrix}. \tag{A.4}$$

Let $Z_n := [\ln F(t, t+\tau_1), \ldots, \ln F(t, t+\tau_M)]'$ represent log futures prices, where $\tau_1, \ldots, \tau_M$ are the *times to maturity* for these $1, \ldots, M$ futures contracts. From (24) the *measurement equation* becomes

$$\mathbf{Z}_n = C_n X_n + d_n + \boldsymbol{\varepsilon}_n, \tag{A.5}$$

where

$$C_n = \begin{bmatrix} e^{-k_x\tau_1} & e^{-k_y\tau_1} & 1 \\ \vdots & \vdots & \\ e^{-k_x\tau_M} & e^{-k_y\tau_M} & 1 \end{bmatrix}, \tag{A.6}$$

$$d_n = \begin{bmatrix} \lambda_x(e^{-k_x\tau_1}-1) + \lambda_y(e^{-k_y\tau_1}-1) + (u-\lambda_p)\tau_1 \\ +\frac{1}{2}\left(\begin{matrix} \frac{1-e^{-2k_x\tau_1}}{2k_x}\sigma_x^2 + \frac{1-e^{-2k_y\tau_1}}{2k_y}\sigma_y^2 + \sigma_p^2\tau_1 \\ +2\frac{1-e^{-(k_x+k_y)\tau_1}}{k_x+k_y}\rho_{xy}\sigma_x\sigma_y + 2\frac{1-e^{-k_x\tau_1}}{k_x}\rho_{xp}\sigma_x\sigma_p + 2\frac{1-e^{-k_y\tau_1}}{k_y}\rho_{yp}\sigma_y\sigma_p \end{matrix} \right) \\ \vdots \\ \lambda_x(e^{-k_x\tau_M}-1) + \lambda_y(e^{-k_y\tau_M}-1) + (u-\lambda_p)\tau_M \\ +\frac{1}{2}\left(\begin{matrix} \frac{1-e^{-2k_x\tau_M}}{2k_x}\sigma_x^2 + \frac{1-e^{-2k_y\tau_M}}{2k_y}\sigma_y^2 + \sigma_p^2\tau_M \\ +2\frac{1-e^{-(k_x+k_y)\tau_M}}{k_x+k_y}\rho_{xy}\sigma_x\sigma_y + 2\frac{1-e^{-k_x\tau_M}}{k_x}\rho_{xp}\sigma_x\sigma_p + 2\frac{1-e^{-k_y\tau_M}}{k_y}\rho_{yp}\sigma_y\sigma_p \end{matrix} \right) \end{bmatrix}, \tag{A.7}$$

and ε_n is an error term allowing noise in the sampling of data with covariance matrix†

$$H = \begin{bmatrix} \xi_1^2 & 0 & 0 \\ \vdots & \vdots & \\ 0 & 0 & \xi_M^2 \end{bmatrix}. \tag{A.8}$$

† See previous footnote.